ERRATA: GRAPHICAL COMBINATORICS AND A DISTRIBUTIVE LAW FOR MODULAR OPERADS

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ABSTRACT. This note describes corrections to two mistakes in the published version of "*Graphical combinatorics and a distributive law for modular operads*" (*Adv. Math.*, Volume 392, 2021, [2]): I first correct a small error in Section 6. The rest of the note is concerned with an error that occurs in Section 4.1. Whilst not serious for the mathematics, it does cause some issues with terminology.

SMALL MISTAKE IN SECTION 6

On [2, page 61], there is a sentence that begins "*But then* $\mathsf{el}(\bigcirc) \cong \mathsf{el}(\shortmid)$*, and hence* $S(\bigcirc) \cong S(\shortmid)$ ..." ([2, page 61]). This should simply read "*But this would imply that* $S(\bigcirc) \cong S(\shortmid)$ ...". The rest of the section is unchanged.

EXPLANATION OF THE ERROR IN SECTION 4.1

In Proposition 4.8, I incorrectly identified the morphisms in the graph category $\mathsf{Grpsh}_\mathsf{f}(\mathtt{D})$ that satisfy the conditions of Lemma 4.7 with the monomorphisms in $\mathsf{Grpsh}_\mathsf{f}(\mathtt{D})$. Throughout the rest of the paper, I then used the term 'monomorphism' to refer to these morphisms. However, whilst all monomorphisms in $\mathsf{Grpsh}_\mathsf{f}(\mathtt{D})$ satisfy the conditions of Lemma 4.7, the converse does not hold. Hence Proposition 4.8, and my terminology, are incorrect.

Whilst the error in Proposition 4.8 – and the subsequent misuse of the term 'monomorphism' – is conspicuous and embarrassing, it does not have mathematical consequences for the main results: Throughout the paper, I merely use the terminology of 'monomorphism' to refer to graph morphisms satisfying the three conditions in Lemma 4.7, but never the actual property of being a monomorphism.

Indeed, there are several instances throughout the paper, including Examples 3.22, 3.31 and Equations (4.12) and (4.13) that are used to obtain the main results but are actually counter-examples to 4.8.

Hackney, Robertson and Yau, in [1, Section 1.3] (see, in particular, their Example 1.23) explain explicitly that the category of morphisms in $\mathsf{Grpsh}_\mathsf{f}(\mathtt{D})$ that satisfy the conditions of Lemma 4.7 is larger than the category of monomorphisms in $\mathsf{Grpsh}_\mathsf{f}(\mathtt{D})$. They introduced the term 'embedding' to refer to morphisms that satisfy the conditions of Lemma 4.7. I adopt this terminology for the revisions below.

REVISIONS TO SECTION 4.1

The error occurs in Section 4.1. This is currently titled "*Pullbacks and monomorphisms in* $\mathsf{Grpsh}_\mathsf{f}(\mathtt{D})$" but should be renamed "*Pullbacks and embeddings in* $\mathsf{Grpsh}_\mathsf{f}(\mathtt{D})$".

The content from the paragraph "*Epimorphisms in* $\mathsf{Grpsh}_\mathsf{f}(\mathtt{D})$ *are pointwise surjections ... étale morphisms do not admit a unique epi- mono factorisation.*" before Example 4.6 on p.36 until the end of Section 4.1 on p.37 should be replaced with the following:

Observe that any (possibly empty) graph $\mathcal{H}$ has the form $\mathcal{H}' \amalg \mathcal{S}$ where $\mathcal{H}'$ is a graph without stick components and $\mathcal{S}$ is a shrub.

Definition E.1. *A morphism $f \in \mathsf{Grpsh_f}(\mathtt{D})(\mathcal{H}, \mathcal{G})$ (with $\mathcal{H} = \mathcal{H}' \amalg \mathcal{S}$ as above) is called an* embedding *if the following three conditions hold:*

(i) the images $f(\mathcal{H}')$ and $f(\mathcal{S})$ are disjoint in $\mathcal{G}$;
(ii) the restriction of f to $\mathcal{S}$ is injective;
(iii) f is injective on $V(\mathcal{H})$ and $H(\mathcal{H})$ (but not necessarily on $E(\mathcal{H})$).

This terminology is due to Hackney, Robertson and Yau [1, Section 1.3].

Lemma E.2. *An embedding $f \colon \mathcal{H} \to \mathcal{G}$ is either pointwise injective or, there exists a pair of ports $e_1, e_2 \in E_0(\mathcal{H})$ such that*

- *$\tau_{\mathcal{H}} e_1, \tau_{\mathcal{H}} e_2 \in s(H)$, and hence $e_2 \neq \tau_{\mathcal{H}} e_1$ (where $\tau_{\mathcal{H}}$ is the involution on $E(\mathcal{H})$),*
- *$\tau_{\mathcal{G}} f(e_2) = f(e_1) \in E_\bullet(\mathcal{G})$ so $\{f(e_1), f(e_2)\}$ forms a $\tau_{\mathcal{G}}$-orbit of inner edges of $\mathcal{G}$.*

Proof. Let $f \in \mathsf{Grpsh_f}(\mathtt{D})(\mathcal{G}, \mathcal{H})$ and assume that e and e' are edges of $\mathcal{H}$ such that $f(e) = f(e')$. If f is an embedding, then either e or e' is a port, since otherwise $e = s(h)$ and $e' = s(h')$, so $f(h) = f(h')$. Moreover, since $f(\tau e) = f(\tau e')$, either τe or $\tau e'$ is a port by the same argument.

Assume therefore, that e is a port. If τe is a port, then e and τe define a stick component of $\mathcal{H}$ and so f violates either condition (i) or condition (ii) of Definition E.1.

So, if f is an embedding, then either $e, \tau e' \in E_0$ and $\tau e, e' \in E_\bullet$, or $e, \tau e' \in E_\bullet$ and $\tau e, e' \in E_0$. In particular $f(e) = f(e')$ and $f(\tau e) = f(\tau e')$ are inner edges of $\mathcal{G}$.

□

Remark E.3. Monomorphisms in $\mathsf{Grpsh_f}(\mathtt{D})$ are pointwise injective morphisms and hence embeddings. For, if $f \colon \mathcal{H} \to \mathcal{G}$ is an embedding such that $e_1, e_2 \in E_0(\mathcal{H})$ are ports and $f(e_1) = f(\tau e_2) \in E_\bullet(\mathcal{G})$, then $f \circ ch_{e_1} = f \circ ch_{\tau e_2} \colon (\shortmid) \to \mathcal{G}$ and hence f is not a monomorphism in $\mathsf{Grpsh_f}(\mathtt{D})$.

Example E.4. Let $\mathcal{W}$ be the wheel graph with vertex $v \in V(\mathcal{W})$. The essential morphism $\iota_v \colon \mathcal{C}_{\mathbf{v}} \to \mathcal{W}$ is a pointwise surjective embedding. In fact, for all $k \geq 1$, the canonical morphism $\mathcal{L}^k \to \mathcal{W}^k$ (Example 3.31) is an epimorphic embedding that is not a monomorphism.

For all finite sets X and Y, the canonical étale morphisms $\mathcal{C}_{X \amalg \{x_0\}} \amalg \mathcal{C}_{Y \amalg \{y_0\}} \to \mathcal{M}^{X,Y}_{x_0,y_0}$ and $\mathcal{C}_{X \amalg \{x_0, y_0\}} \to \mathcal{N}^X_{x_0,y_0}$ are epimorphic embeddings but not monomorphisms.

Remarks on revisions to Section 4.1. In summary, the changes to Section 4.1 are:

- the title is modified;
- Definition E.1 and Lemma E.2 replace Lemma 4.7;
- Proposition 4.8 is deleted altogether;
- Examples 4.6 and 4.9 are corrected and combined into Example E.4.

The revised text is shorter and simpler. This is because the discussion of monomorphisms is not needed in the rest of the paper.

It may be noted however, that embeddings in the wide subcategory of $\mathsf{Grpsh_f}(\mathtt{D})$ whose objects are graphs without stick components are monomorphisms in this subcategory. (In fact, in an early draft of the paper, embeddings were more correctly called *weak subgraphs* or *weak monos*. This provides an indication of why the further consequences of the error are minor.)

Further consequences

The main revision is that from Section 4.2 on, every use of the term 'monomorphism' in the context of graph morphisms should be replaced with the term 'embedding'.

In the remainder of this note, I review every instance where this occurs, as well as all results and examples that, either directly or indirectly, reference Proposition 4.8, and explain that there are no further mathematical consequences of the error in Proposition 4.8.

0.1. **Section 4.2: Graph neighbourhoods and the essential category $\mathsf{es}(\mathcal{G})$.** Replace every occurrence of the term 'monomorphism' in this section – including in Definitions 4.10 and 4.15 and in Lemma 4.14 – with 'embedding'.

The discussion of neighbourhoods from the beginning of Section 4.2 until Lemma 4.14 only uses the defining characteristics of embeddings, and does not require any map to be a monomorphism.

Remark E.5. In fact, the discussion around (4.12) and (4.13) explicitly uses the fact that the maps $i_{\widehat{\mathcal{I}}} \colon \mathcal{G}_{\widehat{\mathcal{I}}} \to \mathcal{G}$ discussed there are not monomorphisms.

For example, the statement
"Moreover, any surjective embedding $\mathcal{G}' \to \mathcal{G}$ factors as $\mathcal{G}' \xrightarrow{\cong} \mathcal{G}_{\widehat{\mathcal{I}}} \xrightarrow{i_{\widehat{\mathcal{I}}}} \mathcal{G}$ for some unique $\mathcal{I} \subset \mathcal{S}(E_\bullet)$"
may be observed as follows: Let $f \colon \mathcal{G}' \to \mathcal{G}$ be a a surjective embedding. Then, by Lemma E.2, f is either bijective on edges or there are mutually distinct pairs of ports $\{e_1^1, \tau e_2^1\}, \dots \{e_1^n, e_2^n\}$ of $\mathcal{G}'$ such that $f(e_1^i) = \tau f(e_2^i), \tau f(e_1^i) = f(e_2^i) \in E_\bullet(\mathcal{G})$ are distinct internal edges of $\mathcal{G}$ for all $1 \leq i \leq n$. These uniquely define the shrub $\mathcal{I}$ and the morphism $\mathcal{G}_{\widehat{\mathcal{I}}} \xrightarrow{i_{\widehat{\mathcal{I}}}} \mathcal{G}$.

In particular, Lemma 4.14 follows as before.

Changing the terminology in Definition 4.15 has no effect on the rest of Section 4.2: It follows from the definition of the essential morphism in Examples 3.28 and 3.29 that $\mathsf{es}(\mathcal{G})$ has no non-trivial isomorphisms, and the proof of Lemma 4.16 is also unchanged.

0.2. **Section 4.3: Boundary-preserving étale morphisms.** Replace every occurrence of the term 'monomorphism' in these sections with 'embedding'.

The first sentence of Proposition 4.18 should read:
"If $u \colon \mathcal{U} \to \mathcal{G}$ is an étale embedding *for which there is a $k \in \mathbb{N}$ such that $f^{-1}(\mathcal{U}) \cong k(\mathcal{U})$, then also $f^{-1}(\mathcal{V}) \cong k(\mathcal{V})$ for all* embeddings *$\mathcal{V} \to \mathcal{U}$."*
The proof uses only Definition E.1 and the universal property of pullbacks.

In the third and last paragraph of the proof of Proposition 4.22, there are references to Proposition 4.8. These should be replaced with references to Lemma E.2 since it is this result that is used. The rest of the proof of Proposition 4.22 only uses the defining properties of embeddings.

Remark E.6. It should be noted that the proof of Proposition 4.22 is the only place where Proposition 4.8 is referenced. Proposition 4.22 is, itself, further used in the proof of Proposition 7.17. This result also makes no use of the properties of monomorphisms. (Likewise in the discussion at the beginning of Section 7.4, which references Proposition 7.17.)

0.3. **Sections 4.4 and 4.5.** Replace every occurrence of the term 'monomorphism' in these sections with 'embedding'.

The proof of Proposition 4.32 – which is not included in the text – is unaffected by this change, since it relies only on Definition E.1 and Lemma E.2.

0.4. **Section 5: Non-unital modular operads.** There are two uses of the term 'monomorphism' in Section 5. Both should be replaced with the term 'embedding'.

The last sentence of the proof of Lemma 5.11 should now read:
"It follows that $\overline{p}\colon \mathcal{G} \to \overline{\mathcal{G}}$ *is an étale* embedding*, and the lemma is proved."*
This follows immediately from the previous sentence and Definition 5.10 of gluing data.

The second to last sentence of the statement of Corollary 5.18 should now read:
" For each $(\mathcal{C}, b) \in \mathsf{el}(\mathcal{G})$*, the universal map* $\mathbf{\Gamma}(b) \to \mathbf{\Gamma}(\mathcal{G})$ *is an étale* embedding*."*
There is no change to the proof.

0.5. **Section 8: A nerve theorem for modular operads.** There are two uses of the term 'monomorphism' in Section 8. Both should be replaced with the term 'embedding'.

The sentence before Lemma 8.7, should now read:
"For each $(\mathcal{C}, b) \in \mathsf{el}(\mathcal{G})$*, let* $\iota^i_b\colon \mathbf{\Gamma}^i(b) \to \mathbf{\Gamma}^i(\mathcal{G})$ *denote the defining* embedding*."*
This follows from Corollary 5.18 (corrected form). In the subsequence proofs, no use is made of the maps $\iota^i_b\colon \mathbf{\Gamma}^i(b) \to \mathbf{\Gamma}^i(\mathcal{G})$ having the property of being monomorphisms.

The correct description of Hackney, Robertson and Yau's category in Section 8.4 on Weak modular operads should read:
"The category U *does not contain those morphisms in* $\mathsf{CetGr}_* \hookrightarrow \Xi$ *that factor through* $z\colon \mathcal{C}_{\mathbf{0}} \to (\shortmid)$ *or* $\kappa^m\colon \to (\shortmid)$*,* $m \geq 1$*, nor does it contain any morphisms of* CetGr *that are not* embeddings*."*

References

[1] Philip Hackney, Marcy Robertson, and Donald Yau. A graphical category for higher modular operads. *Adv. Math.*, 365:107044, 2020.

[2] Sophie Raynor. Graphical combinatorics and a distributive law for modular operads. *Adv. Math.*, 392:Paper No. 108011, 87, 2021.